SHORT COMMUNICATIONS

One-dimensional unsteady general solution for Wang bioheat transfer equation with heat source proportional to temperature*

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Abstract An analytical general solution is derived for the non-Fourier Wang bioheat transfer model with special internal heat production. It is valuable for finding the special solutions with specified conditions and for expanding the understanding of the non-Fourier heat conduction phenomena in living tissues, for example, the controversial temperature fluctuation effects. This analytical solution can also be used as the benchmark solution to check the numerical calculations and to develop various numerical computational approaches. Because there is an arbitrary function in the expression of internal heat production, this solution can actually be applied to many types of internal heat production distributions.

Keywords: **bioheat transfer model, non-fourier effect, analytical solution**

Bioheat transfer models are the theoretical basements for the quantitative analysis and calculation of the heat transfer processes in living tissues. Since the establishment of Pennes equation in 1948, many improved models based on it have been developed by researchers all over the world. The porous medium model proposed by Wang et al.[1] is one of the representatives among them. It is more general since its derivation has no relation with Darcy Law and its application is not confined to Newtonian fluid. A steady general solution was deduced by Wang et al.[2] An unsteady analytical special solution was derived by Cai et al.[3] In addition, an analytical special solution for the non-Fourier Wang equation was reported by Cai et al. in Ref. [4]. Based on the work of Ref. [4], this paper presents an analytical general solution for the non-Fourier Wang equation with special internal heat production to further expand the understanding of highly complex Bioheat transfer phenomena.

The one-dimensional form of the non-Fourier Wang equation can be expressed as

$$A \frac{\partial^2 \theta}{\partial t^2} + B \frac{\partial \theta}{\partial t} - D \frac{\partial^2 \theta}{\partial x^2} + F \frac{\partial^2 \theta}{\partial t \partial x} + G \frac{\partial \theta}{\partial x} = \tau \frac{\partial Q}{\partial t} + Q, \quad (1)$$

where θ is temperature, t is time and x is geometric coordinate. In addition, $A = \rho c \tau$, $B = \rho c$, $D = k$, $F = \rho_b c_b v_x \tau$ and $G = \rho_b c_b v_x$, where ρ is the density of tissue, c the specific heat of tissue, τ the heat relaxation time, k the thermal conductivity, ρ_b the density of blood, c_b the specific heat of blood and v_x is the blood velocity. These coefficients are assumed to be constant as in Ref. [4] to simplify the derivation procedures. Moreover, $Q = q_m + q_r$, where q_m and q_r are the metabolic heat and heat production inside the biotissue by the external factors, respectively.

The basic equation (1) is a second-order partial differential equation with the dependent variable θ. Q is commonly a given function. To acquire the analytical solution of Eq. (1), the following simple relationship is assumed

$$Q = E + C\theta, \quad (2)$$

where C is an undetermined coefficient. Then the equation (1) becomes

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$$A\frac{\partial^2\theta}{\partial t^2}+B\frac{\partial\theta}{\partial t}-D\frac{\partial^2\theta}{\partial x^2}+F\frac{\partial^2\theta}{\partial t\partial x}+G\frac{\partial\theta}{\partial x}=\tau C\frac{\partial\theta}{\partial t}+C\theta+E. \qquad (3)$$

According to the formal characteristic of equation (1), and the authors' experiences in solving heat conduction partial differential equations[5,6], an analytical general solution with special internal heat production can be obtained as follows:

$$\theta=\exp\left(-\frac{t}{\tau}\right)\left[\varphi_1\left(x+\frac{\sqrt{F^2+4AD}-F}{2A}t\right)+\varphi_2\left(x-\frac{\sqrt{F^2+4AD}+F}{2A}t\right)\right]+\frac{\tau}{\rho c}E, \qquad (4)$$

$$Q=E-\frac{\rho c}{\tau}\theta, \qquad (5)$$

where φ_1 and φ_2 are arbitrary differentiable functions. The correctness of this analytical solution can be proven easily by substituting equations (4) and (5) into equation (1) and so do the solutions below. The solution (4) and (5) depicts the temperature distribution in biotissue medium for any boundary conditions and initial conditions reflected by the non-Fourier Wang equation with special internal heat production.

It is noticeable that the solution includes various possible forms of temperature fluctuation effects. For example, it includes the superposition of numerous sine heat waves with arbitrary amplitudes and arbitrary periodicities, as the special solution (6) and (5) below demonstrated, in which K_i is an arbitrary constant.

$$\theta=\exp\left(-\frac{t}{\tau}\right)\cdot\left[K_1\sum_{K_2}\sin K_2\left(x+\frac{\sqrt{F^2+4AD}-F}{2A}t\right)+K_3\sum_{K_4}\sin K_4\left(x-\frac{\sqrt{F^2+4AD}+F}{2A}t\right)\right]+\frac{\tau}{\rho c}E, \qquad (6)$$

$$Q=E-\frac{\rho c}{\tau}\theta. \qquad (5)$$

This is a typical phenomenon of the non-Fourier heat transfer effect, based on the hypothesis of heat disturbance propagation speed being finite. We have reported an analytical special solution with heat wave effects for the Chen-Holmes equation[7], which is another typical perfusion heat transfer model. The relationship between thermal parameters $\rho c=\tau w_b c_b$ is required for the existence of that solution, where w_b is the blood perfusion rate, while the total internal heat production is an arbitrary constant. We primarily discussed the physiological meaning of that solution and its explicit criterion to consider the existence of non-Fourier heat wave. The criterion was compared with that proposed by Liu et al.[8] The similarity in modality and difference in criterion between Wang equation and Chen-Holmes equation in reflecting heat wave effect may be meaningful. However, there are still disputes about the prime reason causing temperature fluctuation effects in biotissue (one opinion being that the response of blood perfusion rate is the main reason). Therefore the temperature wave phenomena depicted by the solution (6) and (5) are not further discussed in this note. Nevertheless, it is necessary to introduce the non-Fourier heat transfer law into the original Wang equation, since the hypothesis of infinite heat disturbance propagation speed is approximate and inadequate. This means the object equation (1) of this note is reasonable.

Eq. (5), as a constraint equation, strictly determines the variation mode of internal heat production. However, utilizing a simple source term balance method[4], it can be deduced that

$$\theta=\exp\left(-\frac{t}{\tau}\right)\left[\varphi_1\left(x+\frac{\sqrt{F^2+4AD}-F}{2A}t\right)+\varphi_2\left(x-\frac{\sqrt{F^2+4AD}+F}{2A}t\right)\right]+\frac{\tau}{\rho c}E, \qquad (4)$$

$$Q=E-\frac{\rho c}{\tau}\theta+\varphi_3(x)\exp\left(-\frac{t}{\tau}\right), \qquad (7)$$

is also a general solution of equation (1) with special internal heat production, in which $\varphi_3(x)$ is an arbitrary function of x. The difference of solution (4) and (7) from the solution (4) and (5) is that the internal heat production Q has a larger function space, and its variation mode can be different with θ to some extent.

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